BAULKHAM HILLS HIGH SCHOOL

JUNE 2012

YEAR 12 ASSESSMENT TASK 3

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in questions 1- 4
- Marks may be deducted for careless or badly arranged work
- Standard integrals are provided on page 7

Total marks – ~~46~~ 44

Exam consists of 7 pages.

Question 1 | **Marks**

a) Find the indefinite integrals:

i) $\int \frac{\tan^{-1} x}{1+x^2} dx$ — 2

ii) $\int \frac{dx}{\sqrt{x^2+2x+5}}$ — 2

iii) $\int x\sqrt{x+1}\ dx$ — 3

b) i) If $\frac{7x^2-3x+2}{(x-2)(x^2+x+2)} = \frac{A}{X-2} + \frac{Bx+C}{x^2+x+2}$ — 2

Find A, B and C

ii) Hence determine — 2

$$\int \frac{7x^2-3x+2}{(x-2)(x^2+x+2)} dx$$

c) Evaluate — 3

$$\int_0^{\frac{\pi}{2}} \frac{4\ d\theta}{3+5cos\theta}$$

Question 2 - Start a new page **Marks**

a) Let n be a positive integer and let

$$I_n = \int_1^3 (\ln x)^n \, dx$$

i) Prove that $I_n = 3(\ln 3)^n - nI_{n-1}$ **2**

ii) Hence determine

$$\int_1^3 (\ln x)^2 dx$$ **3**

b) The base of a solid is a parabolic segment of $y = x^2$ cut off by the chord $y = 4$ **5**

Each plane sector of the solid perpendicular to the axis of the parabola is a rectangle whose base is a chord and its height is $\frac{1}{2}(4 - y)$. Find its volume.

Question 3 - **Start a new page** **Marks**

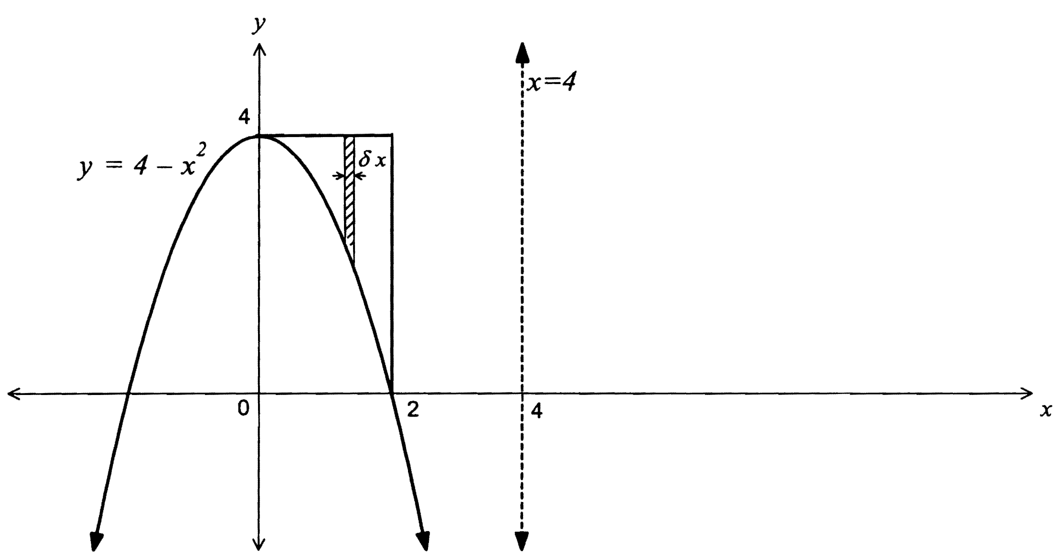
a) i) Prove that

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$ **2**

ii) Use this to evaluate

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$$ **4**

b) The area bounded by $y = 4 - x^2$, $x = 2$ and $y = 4$ is rotated about the line $x = 4$
Using the method of cylindrical solids



i) Show that the volume of a cylindrical shell of thickness δx is

$$\pi x^2(8-2x)\delta x$$ **2**

ii) Find the volume of the solid. **3**

Question 4 - Start a new page **Marks**

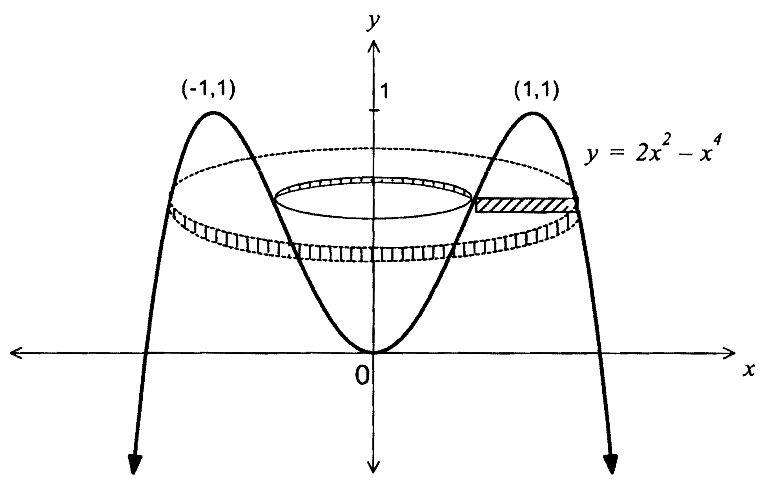
a) i) Show that

$$\int_1^2 \frac{1}{x^2}\ln(x+1)\,dx = \frac{1}{2}\ln\frac{4}{3} + \int_1^2 \frac{1}{x(x+1)}\,dx$$ **2**

ii) Hence evaluate

$$\int_1^2 \frac{1}{x^2}\ln(x+1)\,dx \quad \text{in its simplest form}$$ **3**

b) The region bounded by the curve $y = 2x^2 - x^4$ and $y = 0$ is rotated about the $y-axis$, as shown in the diagram below.



i) Show that the cross sectional area of a slice perpendicular to the $y - axis$ is **2**

$$A = 2\pi\sqrt{1-y}$$

ii) Find the volume formed. **2**

~ End of Exam ~

MATHEMATICS EXT 2 JUNE 2012.

1-14
2-10
3-11
4-9

/44

a) i) $\int \frac{\tan^{-1} x}{1+x^2}\,dx$

let $u = \tan^{-1} x$

$\therefore du = \frac{1}{1+x^2}\,dx$

$I = \int u\,du$

$= \frac{u^2}{2} + c$

$= \frac{1}{2}(\tan^{-1} x)^2 + c$ (2)

ii) $\int \frac{dx}{\sqrt{x^2+2x+5}}$

$= \int \frac{dx}{\sqrt{(x+1)^2+4}}$

$= \ln\left(x+1+\sqrt{x^2+2x+5}\right) + c$ (2)

iii) $\int x\sqrt{x+1}\,dx$

let $u = x+1$

$\therefore du = dx$

$\therefore I = \int (u-1)u^{\frac{1}{2}}\,du$

$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du$

$= \frac{2u^{\frac{5}{2}}}{5} - \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c$

or $= \frac{2}{5}(x+1)^2\sqrt{x+1} - \frac{2}{3}(x+1)\sqrt{x+1} + c$ (3)

b. i) $7x^2 - 3x + 2 \equiv A(x^2+x+2) + (Bx+C)(x-2)$

$x = 2 \Rightarrow 8A = 24$

$\therefore A = 3$ *

coeff of $x^2 \Rightarrow A + B = 7$

$\therefore B = 4$ *

const term $2A - 2C = 2$

$C = 2$ * (2)

ii) $\therefore I = \int \frac{3}{x-2} + \frac{4x+2}{x^2+x+2}\,dx$

$= 3\ln(x-2) + \int \frac{2(2x+1)}{x^2+x+2}\,dx$ (2)

$= 3\ln(x-2) + 2\ln(x^2+x+2) + C$.

c) $\int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos\theta}\,d\theta$ let $t = \tan\frac{\theta}{2}$

$= \int_0^1 \frac{4}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$ $d\theta = \frac{2\,dt}{1+t^2}$

$\theta = 0 \quad t = 0$

$= \int_0^1 \frac{8}{8-2t^2}\,dt$ $\theta = \frac{\pi}{2} \quad t = 1$

$= \int_0^1 \frac{4}{4-t^2}\,dt$

$= \int_0^1 \frac{1}{2-t} + \frac{1}{2+t}\,dt$ (3)

$= \left[-\ln(2-t) + \ln(2+t)\right]_0^1$

$= \ln 3$.

a) i) $I_n = \int_1^3 (\ln x)^n \, dx$

let $u = (\ln x)^n$ $\quad v' = 1$

$u' = \frac{n(\ln x)^{n-1}}{x}$ $\quad v = x$ $\quad$ 1

$\therefore I_n = \left[x(\ln x)^n\right]_1^3 - n\int_1^3 \frac{1}{x}(\ln x)^{n-1} \cdot x \, dx$ $\quad$ (2)

$I_n = 3(\ln 3)^n - n I_{n-1}$ $\quad$ 1

ii) $I_2 = \int_1^3 (\ln x)^2 \, dx$

$I_2 = 3(\ln 3)^2 - 2 I_1$

$I_1 = 3\ln 3 - I_0$ $\quad$ 1

$I_0 = \int_1^3 dx$

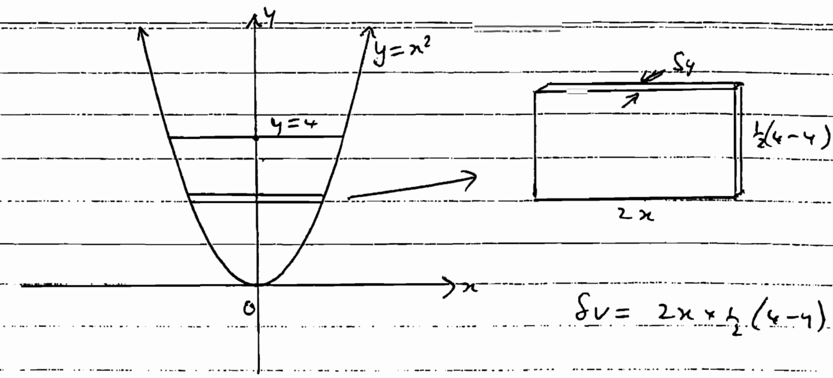
$= 2$ $\quad$ 1

$\therefore I_1 = 3\ln 3 - 2$

$I_2 = 3(\ln 3)^2 - 2(3\ln 3 - 2)$ $\quad$ (3)

$= 2(\ln 3)^2 - 6\ln 3 + 4$ $\quad$ 1

2b)



$\delta V = 2x \times \frac{1}{2}(4-y) \cdot \delta y$

$= x(4-y)\,\delta y$ $\quad$ 1

$V = \sum_{y=0}^{y=4} x(4-y)\,\delta y$ $\quad$ 1

$= \int_0^4 x(4-y)\,dy$

$= \int_0^4 y^{\frac{1}{2}}(4-y)\,dy$ $\quad$ 1

$= \int_0^4 \left(4y^{\frac{1}{2}} - y^{\frac{3}{2}}\right)$ $\quad$ (5)

$= \left[\frac{8}{3}y^{\frac{3}{2}} - \frac{2y^{\frac{5}{2}}}{5}\right]_0^4$ $\quad$ 1

$= \frac{64}{3} - \frac{64}{5}$

$= \frac{128}{15}$ u cubes $\quad$ 1

1 a) $I = \int_1^2 \frac{1}{x^2} \ln(x+1)\,dx$ let $u = \ln(x+1)$ $v' = x^{-2}$

$u' = \frac{1}{x+1}$ $v = -x^{-1}$

$\therefore I = -\left[\frac{\ln(x+1)}{x}\right]_1^2 + \int_1^2 \frac{dx}{x(x+1)}$ 1

$= -\left[\frac{\ln 3}{2} - \ln 2\right] + \int_1^2 \frac{dx}{x(x+1)}$

$= \ln 2 - \frac{\ln 3}{2} + \int_1^2 \frac{dx}{x(x+1)}$ (2)

$= \frac{1}{2}\ln 4 - \frac{1}{2}\ln 3 + \int_1^2 \frac{dx}{x(x+1)}$

$= \frac{1}{2}\ln\frac{4}{3} + \int_1^2 \frac{dx}{x(x+1)}$ 1

let $I_2 = \int_1^2 \frac{dx}{x(x+1)}$

$= \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \left[\ln x - \ln(x+1)\right]_1^2$ 1

$= \{\ln 2 - \ln 3\} - \{\ln 1 - \ln 2\}$ 1

$= 2\ln 2 - \ln 3$

(3)

$\therefore I = \frac{1}{2}\ln\frac{4}{3} + 2\ln 2 - \ln 3$

$= 3\ln 2 - \frac{5}{2}\ln 3$ 1

b) $A = \pi(R^2 - r^2)$

where R, r are the roots of equation

$y = 2x^2 - x^4$ where y is const.

$x^4 - 2x^2 + y = 0$

$x^2 = \frac{2 \mp \sqrt{4-4y}}{2}$ 1 or alt way.

$= 1 \mp \sqrt{1-y}$.

$R^2 = 1 + \sqrt{1-y}$ $r^2 = 1 - \sqrt{1-y}$.

$\therefore R^2 - r^2 = 1 + \sqrt{1-y} - 1 + \sqrt{1-y}$ (2)

$= 2\sqrt{1-y}$

$\therefore A = 2\pi\sqrt{1-y}$.   1

$\delta v = 2\pi\sqrt{1-y}\ \delta y$.

$\therefore v = 2\pi \int_0^1 \sqrt{1-y}\ dy$

$= \left[2\pi \cdot -\frac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1$ 1

$= -\frac{4\pi}{3}[0 - 1]$ (2)

$= \frac{4\pi}{3}$ cu units.   1

3 a) i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

let $u = a - x$ $\quad x = a \quad u = 0$

$\therefore du = -dx$ $\quad x = 0 \quad u = a$ $\quad$ 1

$\therefore \int_0^a f(x)\,dx = \int_a^0 -f(a-u)\,du$ $\quad$ (2)

$= \int_0^a f(a-u)\,du$ $\quad$ 1

$= \int_0^a f(a-x)\,dx$

ii) $I = \int_0^\pi \frac{x \sin x}{1+\cos^2 x}\,dx$ $\quad \sin(\pi - x) = \sin x$

$\cos^2(\pi - x) = (-\cos x)^2 = \cos^2 x$

$= \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}$

$\therefore I = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$ $\quad$ 1

$I = \pi \int_0^\pi \frac{\sin x\,dx}{1+\cos^2 x} - \int_0^\pi \frac{x \sin x\,dx}{1+\cos^2 x}$

$\therefore 2\int \frac{x \sin x}{1+\cos^2 x}\,dx = \pi \int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$ $\quad$ 1

$x = \pi \quad \cos \pi = -1$

let $u = \cos x$ $\quad x = 0 \quad \cos \pi = 1$

$du = -\sin x\,dx$

$\therefore 2I = \pi \int_1^{-1} \frac{-du}{1+u^2}$ $\quad$ 1 $\quad$ (4)

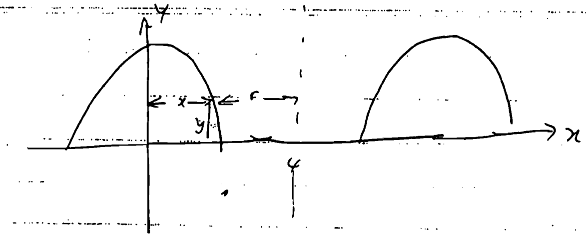
$= \pi \int_{-1}^{1} \frac{du}{1+u^2}$

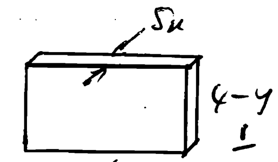
$= \pi \tan^{-1} u$

$= \pi \{\tan^{-1} 1 - \tan^{-1} -1\} = \frac{\pi^2}{2}$

$I = \frac{\pi^2}{4}$ $\quad$ 1

b





1

$\delta v = 2\pi r h$

where $h = y \quad r = 4 - x$.

i) $\therefore \delta v = 2\pi (4-x) y\,\delta x$ $\quad$ (2)

$y = x^2$

$\therefore \delta v = 2\pi x^2 (4-x)\,\delta x$

$= \pi x^2 (8 - 2x)\,\delta x$ $\quad$ 1

ii) $V = \pi \int_0^2 8x^2 - 2x^3\,dx$ $\quad$ 1

$= \pi \left[\frac{8x^3}{3} - \frac{x^4}{2}\right]_0^2$ $\quad$ 1 $\quad$ (3)

$= \pi \left[\frac{64}{3} - 8\right]$

$= \frac{40\pi}{3}$ units^3 $\quad$ 1